To be published in *Journal of Physics A: Mathematical and General* (2002)

On the generalized entropy pseudoadditivity for complex systems

Qiuping A. Wang, Laurent Nivanen, Alain Le Méhauté

Institut Supérieur des Matériaux du Mans, 44, Av. Bartholdi, 72000 Le Mans, France

and Michel Pezeril,

Laboratoire de Physique de l'état Condensé, Université du Maine,

72000 Le Mans, France

We show that Abe's general pseudoadditivity for entropy prescribed by thermal equilibrium in nonextensive systems holds not only for entropy, but also for energy. The application of this general pseudoadditivity to Tsallis entropy tells us that the factorization of the probability of a composite system into product of the probabilities of the subsystems is just a consequence of the existence of thermal equilibrium and not due to the independence of the subsystems.

05.20.-y,05.70.-a,05.90.+m

Suppose an isolated system composed of two subsystems 1 and 2. One of the basic assumptions of thermodynamics is the existence of an equilibrium state in the compound system at which $T_1 = T_2$, where T is the absolute temperature. This so called zeroth law is obeyed by Boltzmann statistical mechanics having additive entropy and energy. For nonextensive statistical mechanics (NSM) [1], the validity of this law is a more subtle affair [1–6] which depends on the relationships $aS_{12} = f_{12}(aS_1, aS_2)$ and $bE_{12} = g_{12}(bE_1, bE_2)$, where S is entropy and E the internal energy. The numbers in the indexes of the functions indicate the dependence of the latter. a and b are constants used to make each variable dimensionless in the above equalities. From now on, we let $a = b = 1$ so that S and E become dimensionless.

The zeroth law has been established, in a approximate way [3–6], for NSM with Tsallis entropy and additive energy $E_{12} = E_1 + E_2$. Recently, Abe [7] furthermore find a general pseudoadditivity of entropy required by the existence of thermal equilibrium for additive energy :

$$H(S_{12}) = H(S_1) + H(S_2) + \lambda_S H(S_1)H(S_2), \tag{1}$$

where H is certain differentiable function satisfying $H(0) = 0$ and λ_S a constant depending on the nature of S. For Tsallis entropy [1], H can be proved to be the identity function. Eq.(1) is very interesting because it can be considered as a general criterion of pertinent nonextensive entropies for equilibrium systems and may help to understand Tsallis nonextensive statistical mechanics and to find other nonextensive thermostatistics obeying the zeroth law.

Nonextensive entropy is a consequence of long range correlations or complex (fractal or chaotic) space-time. When interactions are no more limited between or on the walls of the containers of subsystems, energy may be nonadditive. This issue has been widely discussed and energy nonadditivity was clearly shown for some cases with long range interactions [8–16]. So, inspired by Abe's work, we naturally ask the following question : what is the kind of energy nonadditivity that satisfies the requirement of existence of thermal equilibrium? In

this letter, along the line of Abe, we show that the above *equilibrium pseudoadditivity* holds not only for entropy, but also for internal energy in more general cases where energy-type thermodynamic variables are not extensive.

To establish the zeroth law, we make a small variation of the total entropy dS_{12} given by

$$dS_{12} = \frac{\partial f_{12}}{\partial S_1} dS_1 + \frac{\partial f_{12}}{\partial S_2} dS_2, \tag{2}$$

and a variation of total energy E_{12} given by

$$dE_{12} = \frac{\partial g_{12}}{\partial E_1} dE_1 + \frac{\partial g_{12}}{\partial E_2} dE_2. \tag{3}$$

At thermal equilibrium, $dS_{12} = 0$ should hold and $dE_{12} = 0$ results from energy conservation, leading to

$$\frac{\partial f_{12}}{\partial S_1} \frac{\partial S_1}{\partial E_1} dE_1 = -\frac{\partial f_{12}}{\partial S_2} \frac{\partial S_2}{\partial E_2} dE_2, \tag{4}$$

and

$$\frac{\partial g_{12}}{\partial E_1} dE_1 = -\frac{\partial g_{12}}{\partial E_2} dE_2. \tag{5}$$

So we obtain

$$\frac{\frac{\partial f_{12}}{\partial S_1}}{\frac{\partial g_{12}}{\partial E_1}} \frac{\partial S_1}{\partial E_1} = \frac{\frac{\partial f_{12}}{\partial S_2}}{\frac{\partial g_{12}}{\partial E_2}} \frac{\partial S_2}{\partial E_2}. \tag{6}$$

Equilibrium means that this above equation yields the zeroth law which is in general given by following equality :

$$F_1 = F_2 \tag{7}$$

where F_1 and F_2 are the same functions depending on subsystem-1 and -2, respectively. This constraint from the zeroth law implies following factorizations of the derivatives in Eq.(6) :

$$\frac{\partial f_{12}}{\partial S_1} = \phi_{12} \omega_1 \nu_2, \tag{8}$$

$$\frac{\partial f_{12}}{\partial S_2} = \phi_{12}\nu_1\omega_2, \tag{9}$$

$$\frac{\partial g_{12}}{\partial E_1} = \theta_{12}\mu_1\upsilon_2, \tag{10}$$

and

$$\frac{\partial g_{12}}{\partial E_2} = \theta_{12}\upsilon_1\mu_2. \tag{11}$$

where ϕ, ω, ν, θ, υ and μ are certain functions of the subsystems indicated by the indexes. From Eq.(6) we can write

$$\xi_1\frac{\partial S_1}{\partial E_1} = \xi_2\frac{\partial S_2}{\partial E_2}. \tag{12}$$

with $\xi_1 = \frac{\omega_1\upsilon_1}{\nu_1\mu_1}$ and $\xi_2 = \frac{\omega_2\upsilon_2}{\nu_2\mu_2}$. This shows that Eq.(8) to (11) are really the most general forms of the derivatives of f and g satisfying Eq.(7).

We find that all the calculations of Abe [7] hold for f as well as for g. So we can replace S by E in Eq.(1). For a system containing N subsystems in equilibrium, we have

$$\ln[1+\lambda_x H_x(x_{12..N})] = \sum_{i=1}^{N}\ln[1+\lambda_x H_x(x_i)]. \tag{13}$$

where x can be entropy S or energy E. In general, H_S and λ_S are different from H_E and λ_E, respectively.

We mention here as example two nonextensive cases with Tsallis entropy ($S = -\frac{1-\mathtt{Tr}\rho^q}{1-q}$, ρ is density operator) where the zeroth law is claimed to be verified.

1. In the Tsallis nonextensive statistics with escort probability, H_S and H_E are the identity function, $\lambda_S = (1-q)$ and $\lambda_E = 0$ [1,3–6] (i.e. E is extensive and $S_{12} = S_1 + S_2 + (1-q)S_1S_2$).

2. Another possible case is with Tsallis entropy combined with a so called incomplete normalization [17–19] where H_x is the identity function and $\lambda_x = q-1$ for both entropy and energy (i.e. $x_{12} = x_1 + x_2 + (q-1)x_1x_2$ or, according to Eq.(13), $x_{12..N} = \frac{[\prod_{i=1}^{N}(1+\lambda_x x_i)-1]}{\lambda_x}$ with N subsystems. In this case, the zeroth law can hold without approximation, making it possible to establish an exact nonextensive thermodynamics.

An example of systems satisfying pseudoadditivity of entropy is given by Abe [7] with $H_x(x) = \sqrt{x}$ for black hole entropy proportional to its horizon area. This discussion should hold for energy as well according to the first law of thermodynamics for black hole [20] if electromagnetic work is absent. So it would indeed be interesting to study black hole within a generalized thermodynamics with nonadditive entropy and energy as well in view of the difficulty with Boltzmann statistics due to the presence of thermal (infrared) divergence [21]. Another example can be given with the long-range ferromagnetic spin model of which the internal energy is given by $E(N,T) = c(T)N\frac{N^{1-\alpha/d}-1}{1-\alpha/d}$ [8,10,12] where N is the number of particles in the model supposed additive, d the dimension of space, α the exponent in the factor $1/r^\alpha$ of the long range potential [8–14] and c(T) certain function of temperature T. When $N \to \infty$ and $\alpha > d$, $E(N,T) = c(T)N$ is additive. On the other hand, when $\alpha = d$, $E(N,T) = c(T)N \ln N$. If we put, e.g., $H_E(E) = \{exp[e^{E/c(T)N}] - e\}/e$ and $\lambda_E = 1$, then E satisfies Eq.(1) or Eq.(13). When $0 < \alpha < d$, $E(N,T) = c(T)N^{2-\alpha/d}$, we can choose $H_E(E) = [\frac{E}{c(T)}]^{1/(2-\alpha/d)}$ and $\lambda_E = 0$ for energy to satisfy Eq.(1) or Eq.(13). There exist other choices, e.g. $H_E(E) = \exp\{[\frac{E}{c(T)}]^{1/(2-\alpha/d)}\} - 1$ and $\lambda_E = 1$.

As a matter of fact, it seems that, for a given explicit relation between energy or entropy and number of subsystems or volume supposed additive (i.e. $V_{12} = V_1 + V_2$ and $N_{12} = N_1 + N_2$), the finding of a function H satisfying Eq.(1) is a trivial affair. The most essential contribution of Abe's work, in our opinion, is that Eq.(1) finally makes it clear that the factorization of the compound probability of a composite system into product of the probabilities of the subsystems is a consequence of the existence of thermodynamic equilibrium *if Tsallis entropy applies*, because H is identity function here. It is straightforward to show that : $S_{12} = S_1 + S_2 + (1-q)S_1S_2 = -\frac{1-\mathtt{Tr}\rho_1^q}{1-q} - \frac{1-\mathtt{Tr}\rho_2^q}{1-q} + (1-q)\frac{1-\mathtt{Tr}\rho_1^q}{1-q}\frac{1-\mathtt{Tr}\rho_2^q}{1-q} = -\frac{1-\mathtt{Tr}(\rho_1\rho_2)^q}{1-q} = -\frac{1-\mathtt{Tr}\rho_{12}^q}{1-q}$, which means $\rho_{12} = \rho_1\rho_2$ [or that $p_{ij}^q(1,2) = p_i^q(1)p_j^q(2)$ implies $p_{ij}(1,2) = p_i(1)p_j(2)$ where $p_i(1)$ or $p_j(2)$ is the probability for subsystem-1 or -2 to be at state i or j and $p_{ij}(1,2)$ is the probability for the composite system to be at the product state ij]. So *the product probability is rather due to thermal equilibrium instead of independence of noninteracting or weakly interacting subsystems as claimed or insinuated in most of the cur-*

rent publications. On this basis, we can finally free ourselves from the paradox of addressing *noninteracting independent systems* with *nonadditive entropy due to long range interactions*. Additive energy based on independent systems with only short range interactions does not conform with the spirit of nonextensive statistical mechanics.

The law of product probability must be respected for any thermodynamic system in equilibrium. This means that, for NSM with the Tsallis q-exponential distribution, nonextensive energy is needed for *exact treatments of nonextensive interacting systems*. In this sense, all treatments within NSM with additive energy should be viewed as a kind of *extensive approximation* and should be proceeded with great care because they give sometimes very different results from the treatment respecting probability factorization or product state [19,22]. This above "equilibrium interpretation" of the factorization hypothesis of compound probability may have important consequences on the applications of NSM to many-body systems. More detailed discussions on this issue are given in others papers of ours [23].

Summing up, we have applied Abe's method of finding general entropy pseudoadditivity to a more general case where both entropy and energy are nonextensive. Under thermal equilibrium, the energy of a nonextensive composite system also obeys Abe-type pseudoadditivity.

We would like to thank Professor Sumiyoshi Abe for critical reading of this manuscript and important comments. Thanks are also due to Professor S. Ruffo for valuable discussions.

---

[1] C. Tsallis, *J. Statis. Phys.,* **52**(1988)479;

C. Tsallis, R.S. Mendes and A.R. Plastino, *Physica A,* **261**(1999)534;

Silvio R.A. Salinas and C. Tsallis, *Brazilian Journal of Physics(special issue: Nonextensive Statistical Mechanics and Thermodynamics),* **29**(1999)1-50

[2] G.R. Guerberoff and G.A. Raggio, *J. Math. Phys.,* **37**(1996)1776;

G.R. Guerberoff, P. A. Pury and G.A. Raggio, *J. Math. Phys.* ,**37**(1996)1790

[3] S. Abe, *Physica A,* **269**(1999)403-409.

[4] S. Abe, *Physica A,* **300**(2001)417;

S. Abe, S. Martinez, F. Pennini and A. Plastino, *Phys. Lett.A,* **281**(2001)126;

S. Abe, *Phys. Lett. A,* **278**(2001)249

[5] S. Martinez, F. Nicolas, F. Pennini, and A. Plastino, *Physica A,* **286**(2000)489, physics/0003098;

S. Martinez, F. Pennini, and A. Plastino, *Phys. Lett. A,* **278**(2000)47;

S. Martinez, F. Pennini, and A. Plastino, *Physica A,* **295**(2001)416;

S. Martinez, F. Pennini, and A. Plastino, *Physica A,* **295**(2001)246

[6] Raul Toral, *On the definition of physical temperature and pressure for nonextensive thermodynamics*, cond-mat/0106060

[7] S. Abe, *Phys. Rev. E,* **63**(2001)061105

[8] C. Tsallis, *Fractals,* **3**(1995)541;

[9] C. Anteneodo and C. Tsallis, *Phys. Rev. Lett.,* **80**(1995)5313;

[10] R. Salazar and R. Toral, *Physica A,* **290**(2001)159;

[11] P. Jund, S.G. Kim and C. Tsallis, *Phys. Rev. B,* **52**(1995)50;

[12] S.A. Cannas and F.A. Tamarit, *Phys. Rev. B,* **54**(1996)R12 661;

[13] L.C. Sampaio, M.P. de Albuquerque, and F.S. de Menezes, *Phys. Rev. B,* **55**(1997)5611;

[14] J.R. Grigera, *Phys. Lett. A,* **217**(1996)47

[15] V. Latora, A. Rapisarda, C. Tsallis, *Physica A,* **305**(2002)129

[16] M. Antoni and S. Ruffo, *Phys. Rev. E,* **52**(1995)2361

[17] Q.A. Wang, Nonextensive statistics and incomplete information, *Euro. Phys. J. B,* **26**(2002)357, cond-mat/0107065

[18] Q.A. Wang, *Chaos, Solitons* & *Fractals,* **12**(2001)1431, cond-mat/0009343

[19] Q.A. Wang, M. Pezeril, L. Nivanen, A. Le Méhauté, *Chaos, Solitons* & *Fractals,* **13**(2001)131, cond-mat/0010294

[20] Sean A. Hayward, *Class. Quant. Grav.,* **15**(1998)3147, gr-qc/9710089

[21] V.P. Frolov and D.V. Fursaev, *Class. Quant. Grav.,* **15**(1998)2041, hep-th/9802010

[22] Lenzi E.K., Mendes R.S., da Silva L.R. and Malacarne L.C., *Physics Lett. A.,* **289**(2001)44

[23] Q.A. Wang, *Quantum distributions prescribed by factorization hypothesis of compound probability, Chaos, Solitons* & *Fractals,* **14**(2002)765, cond-mat/0201248

Q.A. Wang, *Many-body q-exponential distribution prescribed by factorization hypothesis,* Phys. Lett. A (2002), in press, cond-mat/0112211

Q.A. Wang, *Unnormalized nonextensive expectation value and zeroth law of thermodynamics, Chaos, Solitons* & *Fractals,* (2002), in press, cond-mat/0111238

Q.A. Wang, *Extensive form of equilibrium nonextensive statistics, J. Math. Phys.,* (2002), in press, cond-mat/0203448